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20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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I. Introduction

The Wigner Distribution (WD),

$$W_x(t,\omega) = \int_{-\infty}^{\infty} x(t + \tau/2)x^*(t - \tau/2)e^{-j\omega\tau}d\tau \tag{1}$$

of a continuous time signal $x(t)$ is a bilinear signal transformation that is useful for analyzing signals whose frequency content changes with time. The WD originated in the field of quantum mechanics, and has been used to characterize the time-varying signals found in optical communications, speech waveforms and radar/sonar patterns. The ambiguity function of $x(t)$ is related to $W_x(t,\omega)$ via two Fourier transforms. Wigner distributions exhibit several desirable properties: 1) the auto WD of a real or complex signal $x(t)$ is real, 2) $W_x(t,\omega)$ has the same time and frequency support as $x(t)$, and 3) shifts in time or frequency on $x(t)$ produce corresponding shifts in $W_x(t,\omega)$. In addition, the average frequency of $W_x(t,\omega)$, i.e. fixed t_0, is equal to the instantaneous frequency, whereas the average time of $W_x(t,\omega_0)$ is equal to the group delay of the signal. Integration of $W_x(t,\omega)$ with respect to the frequency (time) variable for a fixed time t_0 (frequency ω_0) yields the instantaneous power (energy density) of the signal at $t_0(\omega_0)$. The integral over the entire (t,ω) plane is equal to the signal energy.

$W_x(t,\omega)$ can be viewed as the Fourier Transform of

$$g_x(t,\tau) = x(t + \tau/2)x^*(t - \tau/2) \ , \tag{2}$$

the product of appropriately shifted versions of $x(t)$ and its complex conjugate, $x^*(t)$. When the Fourier integral cannot be calculated in closed form, digital signal processing techniques are often invoked to obtain numerical approximations. A typical signal processing approach is to sample either $g_x(t,\tau)$ or $x(t)$, window the resulting sequence, and then compute its discrete Fourier Transform (DFT). However, $x(t)$ must be sampled at twice the Nyquist rate [2] to avoid aliasing errors when calculating $W_x(t,\omega)$. In many signal processing situations where under-sampling has occurred, anti-aliasing digital filters can be designed to minimize aliasing errors [4,5]. An alternate approach is to view the sequence of samples as part of a spline function approximation to the continuous-time signal $x(t)$. The Fourier integral of a spline approximation can be easily calculated using certain predetermined weighting functions, called diminishing factors [6], to reduce aliasing errors. In many cases, the spline function analysis method produces a smaller approximation error to the Fourier integral than the DFT of the sequence of samples does.

It is the purpose of this research to extend the theory of implicit spline approximations of the Fourier integral to similar approximations of the Wigner distribution. We determine the two-dimensional generalized diminishing operators needed to correct the aliasing errors generated when only an undersampled version of $x(t)$ is available to approximate $W_x(t,\omega)$. This implicit interpolation process "fills-in" the

sampled signal, $x(nT)$, between the signal samples, increases the effective sampling rate, and hence reduces the effects of aliasing when calculating the Wigner distribution. The spline approximation can be calculated very efficiently if one takes advantage of the special structure and symmetry of Wigner distributions.

II. Theoretical Developments

Classen et al [2] and Chan [3] have studied the effects of sampling and windowing $x(t)$ in order to approximate $W_x(t,\omega)$. If $x(t)$ is bandlimited to ω_s, i.e.

$$X(\omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\omega t} dt = 0, \; |\omega| \geq \omega_s = 2\pi f_s \, , \tag{3}$$

then $W_x(t,\omega)$ is also bandlimited to ω_s. The DFT can be used to calculated (3) on a discrete grid of frequency points if $x(t)$ is finite length and sampled every $T \leq \frac{1}{2f_s}$ seconds. However, twice as many samples of $x(t)$, is needed to avoid aliasing errors when calculating $W_x(t,\omega)$ on the same frequency grid, i.e. $T \leq \frac{1}{4f_s}$.

If the signal $x(t)$ has been sampled at the Nyquist rate, then one can avoid aliasing errors in approximating $W_x(t,\omega)$ by interpolating the sampled sequence. However, in addition to the computational expense involved in interpolation algorithms, doubling the number of samples more than doubles the number of calculations needed to approximate $W_x(t,\omega)$. An alternate approach to minimizing the effects of aliasing is to view the undersampled sequence as the equally spaced knots of a

spline approximation to the continuous-time signal x(t).

Let the discrete signal $\{\tilde{x}(n)\}$ represent the uniformly spaced samples of the continuous time signal, x(t). That is,

$$\tilde{x}(n) = x(nT) \ , \qquad n = \ldots, -2, -1, 0, 1, 2, \ldots \ . \tag{4}$$

The Fourier transform of this sampled sequence is

$$\tilde{X}(f) = \sum_{n=-\infty}^{\infty} \tilde{x}(n) e^{-j2\pi fnT} \ . \tag{5}$$

Furthermore, assume $\hat{x}(t)$ is the first-order spline approximation of x(t) that interpolates the samples, x(nT). The Fourier integral of the spline approximation

$$\hat{X}(f) = \int_{-\infty}^{\infty} \hat{x}(t) e^{-j2\pi ft} dt \cong X(f) \tag{6}$$

is easy to compute [6] since

$$\hat{X}(f) = \tilde{X}(f) \ [\frac{\sin\pi fT}{\pi fT}]^2 \tag{7}$$

is the product of the Fourier transform (5) of the signal samples, and the "diminishing factor" $[(\sin\pi fT)/\pi fT]^2$. This diminishing factor is predetermined according to the order of the spline approximation and weights the periodic spectrum of the undersampled sequence, $\tilde{x}(n)$, to produce the unaliased spectrum of the continuous time spline approximation, $\hat{x}(t)$. This technique is said to be "implicit" since the spline function, $\hat{x}(t)$ is never actually computed. Note that since $[(\sin\pi fT)/\pi fT]^2$ is the Fourier transform of the triangular pulse

$$p(t) = \begin{Bmatrix} 1 - |\frac{t}{T}| & , & |t| < T \\ 0 & & \text{otherwise} \end{Bmatrix}, \tag{8}$$

the spline, or piecewise linear, approximation, $\hat{x}(t)$, can be written as the convolution of the discrete sequence $\{\tilde{x}(n)\}$ and the triangular interpolating function, $p(t)$:

$$\hat{x}(t) = \sum_{n=-\infty}^{\infty} \tilde{x}(n)p(t - nT) = \sum_{n=-\infty}^{\infty} x(nT)p(t - nT) \tag{9}$$

In order to apply these results to Wigner distributions, let

$$W_{\hat{x}}(t,\omega) = \int_{-\infty}^{\infty} \hat{x}(t + \tau/2)\hat{x}^*(t - \tau/2)e^{-j\omega\tau}d\tau \cong W_x(t,\omega) \tag{10}$$

be the WD of the spline approximation, $\hat{x}(t)$. Plugging (9) into (10) we obtain

$$W_{\hat{x}}(t,\omega) = \sum_{i=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x(iT)x^*(mT)W_p(t - \frac{m+i}{2} T,\omega) \tag{11}$$

where $W_p(t,\omega)$ is the WD of $p(t)$.

Evaluating (11) at $t = nT$, we find

$$W_{\hat{x}}(nT,\omega) = \sum_{i=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} x[(n + i)T]x^*[(n - m)T]e^{-j\omega(i+m)T}W_p[\frac{m-i}{2} T,\omega] \tag{12}$$

Since $W_p[t,\omega] \equiv 0$, $\forall\ |t| \geq T$, (12) reduces to

$$W_{\hat{x}}(nT,\omega) = \sum_{q=-1}^{1} \{ \sum_{i=-\infty}^{\infty} \tilde{x}(n+i)\tilde{x}((n-q)-i)e^{-j\omega 2iT}\} x$$

$$\{e^{-j\omega qT} W_p[qT/2,\omega]\} \tag{13}$$

Equation (13) is similar in form to (7) since the expression in the first set of brackets is merely the discrete cross Wigner distribution [2] of the undersampled sequences $\tilde{x}(n)$ and $\tilde{x}(n-q)$. Equation (13) could be calculated by appropriately weighting, or diminishing, three cross Wigner distributions. However, a much more efficient method can be obtained by taking advantage of the special symmetry that results when one computes the DFT, with respect to n, of $W_{\hat{x}}(nT,\omega)$.

III. Implementation

Define the convolution sequence (for fixed ω_0)

$$h_{\omega_0}(n) = \sum_{i=-\infty}^{\infty} (\tilde{x}(i)e^{-j\omega_0 iT})(\tilde{x}(n-i)e^{-j\omega_0(n-i)T})^*$$

$$= \sum_{i=-\infty}^{\infty} \tilde{x}(i)\tilde{x}(n-i)e^{-j\omega_0 2iT} e^{j\omega_0 nT} \tag{14}$$

Note that,

$$h_{\omega_0}(2n) = W_{\tilde{x}}(nT,\omega_0) \tag{15}$$

is the discrete WD of the samples $\tilde{x}(n)$ evaluated at the frequency, ω_0.

Let

$$H_{\omega_0}(z) = \sum_{n=-\infty}^{\infty} h_{\omega_0}(n) z^{-n} \tag{16}$$

be the z-transform of $h_{\omega_0}(n)$. Combining (13-14), we obtain

$$W_{\hat{x}}(nT,\omega) = \sum_{q=-1}^{1} h_{\omega}(2n - q) W_p[q\ T/2,\omega] \ . \tag{17}$$

If we define the z-transform

$$\Phi(z,\omega_0) = \sum_{n=-\infty}^{\infty} W_{\hat{x}}(nT,\omega_0) z^{-n} \tag{18}$$

and insert (16-17) into (18), we obtain

$$\Phi(z,\omega_0) = \frac{1}{2} \sum_{q=-1}^{1} \{ \sum_{i=0}^{1} H_{\omega_0}(z^{1/2} e^{-j\pi i}) z^{q/2} e^{-j\pi iq} \} W_p[q\ T/2,\omega_0] \tag{19}$$

Furthermore, if we assume that $\tilde{x}(n)$ is finite length, N, and also assume (without loss of generality) that $\tilde{x}(n)$ is causal, then the DFT can be used to evaluate (16) on a set of 2N discrete normalized frequency points, i.e.

$$H_{\omega_0}(e^{j\frac{2\pi}{2N}k}) = \sum_{n=0}^{2N-1} h_{\omega_0}(n) e^{-j\frac{2\pi}{2N}kn}$$

$$= \tilde{X}\left(\frac{k}{2NT} + \frac{\omega_0}{2\pi}\right) \tilde{X}^*\left(\frac{-k}{2NT} + \frac{\omega_0}{2\pi}\right) \tag{20}$$

where $\tilde{X}(f)$ is the length 2N DFT of the undersampled sequence, $\tilde{x}(n)$ and ω_0 is one of the discrete frequency points $\omega_0 = 0, \pm \frac{2\pi}{2NT}, \pm \frac{2\pi}{2NT}2,$

$\pm \frac{2\pi}{2NT}3$, ... Similarly, (19) can be computed on a NxN grid of normalized frequency points by noting that

$$\Phi(e^{j\frac{2\pi}{N}k},\frac{2\pi}{NT}m) = \frac{1}{2}\sum_{i=0}^{1} H_{\frac{2\pi}{NT}m}(e^{j\frac{2\pi}{2N}k}e^{-j\pi i})\sum_{q=-1}^{1} e^{j\frac{2\pi}{2N}q}e^{-j\pi iq}W_p[q\ T/2,\frac{2\pi}{NT}m]$$

$$= \frac{1}{2}\sum_{i=0}^{1}\{(\tilde{X}(\frac{k+2m-Ni}{2NT})\tilde{X}^*(\frac{-k+2m-Ni}{2NT})\}$$

$$\times\ \{W_p(0,\frac{2\pi}{NT}m) + 2(-1)^i\cos(\frac{2\pi}{2N}k)W_p[T/2,\frac{2\pi}{NT}m]\} \qquad (21)$$

since $W_p(t,\omega) = W_p(-t,\omega)$ for $p(t)$ real and symmetric. Thus, the length 2N DFT of $\tilde{x}(n)$ and the length N vectors $W_p(0,\frac{2\pi}{NT}m)$ and $W_p[T/2,\frac{2\pi}{NT}m]$ need only be computed once and stored in memory. Then for each $m = 0, 1, \ldots, N-1$, the m^{th} column $\Phi(e^{j\frac{2\pi}{2N}k},\frac{2\pi}{NT}m)$, $k = 0, 1, \ldots, N-1$ and the length N inverse DFT,

$$W(nT,\frac{2\pi}{NT}m) = \frac{1}{N}\sum_{k=0}^{N-1}\Phi(e^{j\frac{2\pi}{2N}k},\frac{2\pi}{NT}m)e^{j\frac{2\pi}{N}kn} \qquad (22)$$

are computed using Fast Fourier Transform (FFT) techniques to efficiently obtain the spline approximation of the Wigner Distribution.

IV. Examples

The length N = 41 equal-ripple finite impulse response in Figure 1 was designed using the Remez Exchange algorithm [7]. Its frequency

response, shown in Figure 2 was designed with a unity passband from $F_{P_1} = 0.0$ to $F_{P_2} = 0.15$. The stopband extends from normalized frequency $F_{S_1} = 0.2$ to $F_{S_2} = 0.5$. The passband and stopband errors were equally weighted. Note that the impulse response in Figure 1 was sampled more than twice as fast as required by the Nyquist criterion in order to avoid aliasing errors when calculating its discrete Wigner distribution shown in Figure 3. The WD shown in Figure 4 is that of the impulse response in Figure 1 decimated by a factor of 2. Even though the reduced sampling rate satisfies the Nyquist criterion, it is not fast enough, i.e. it does not meet the requirement that $T \leq \frac{1}{4f_s}$, to avoid aliasing errors in the WD. Finally, the WD of the spline approximation of the decimated impulse response is shown in Figure 5. Note that the effects of aliasing have been removed. The first order spline, or piecewise-linear, approximation did very well approximating the low frequencies and only moderately well approximating the high frequencies, as would be expected. Higher order splines can be used to obtain better approximations.

V. Conclusions

Implicit spline interpolation of a sampled sequence has been used to approximate the Wigner distribution and reduce the effects of aliasing errors that occur when a signal is undersampled. Recall that signals must normally be sampled at twice the Nyquist rate in order to avoid aliasing when calculating its WD. The implicit approximation technique is an efficient alternative to oversampling or explicitly interpolating a given sampled signal. The symmetries resulting in the

double frequency plane in (21) can be utilized to compute both the discrete Wigner distribution and the spline approximation of the WD very efficiently.

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<u>Figure Captions</u>

Figure 1. Impulse response of a length N = 41 equal-ripple lowpass filter designed with unity passband from $F_{P_1} = 0.20$ to $F_{P_2} = 0.15$ and stopband $F_{S_1} = 0.20$ and $F_{S_2} = 0.5$. The sampling interval, T, is assumed to be equal to 1.0.

Figure 2. Frequency response of a length N = 41 equal-ripple lowpass filter designed with unity passband from $F_{P_1} = 0.0$ to $F_{P_2} = 0.15$ and stopband ranging from $F_{S_1} = 0.20$ to $F_{S_2} = 0.5$.

Figure 3. Time (n) vs. frequency (ω) plot of the Wigner distribution (WD) of the impulse response shown in Figure 1. (Sampling interval T = 1.0)

Figure 4. Time (n) vs. frequency (ω) plot of the WD of the decimated lowpass impulse response. (Sampling interval T = 2.0)

Figure 5. Time (n) vs. frequency (ω) plot of the WD of the first-order spline approximation using the decimated lowpass impulse response sequence as the equally spaced spline knots. (Sampling interval T = 2.0).

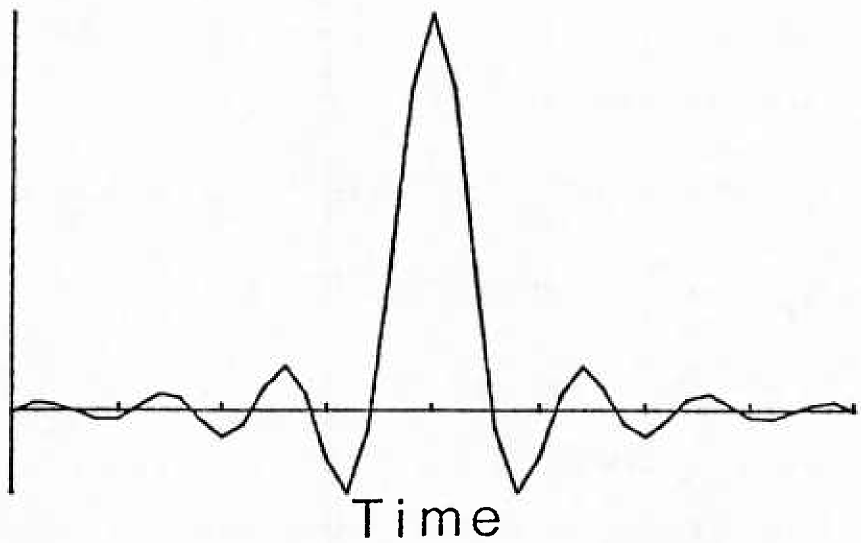

FIGURE 1

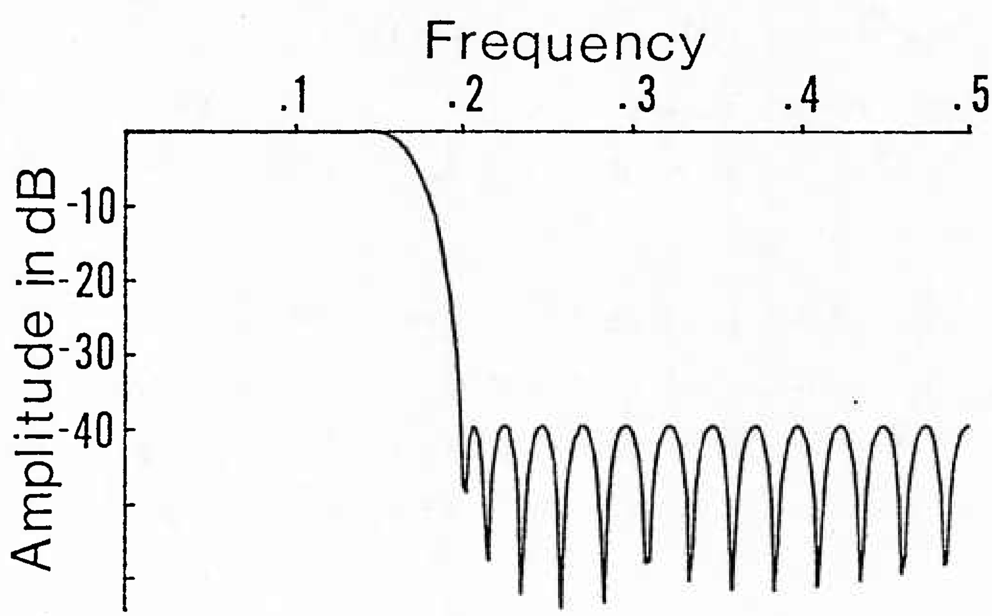

FIGURE 2

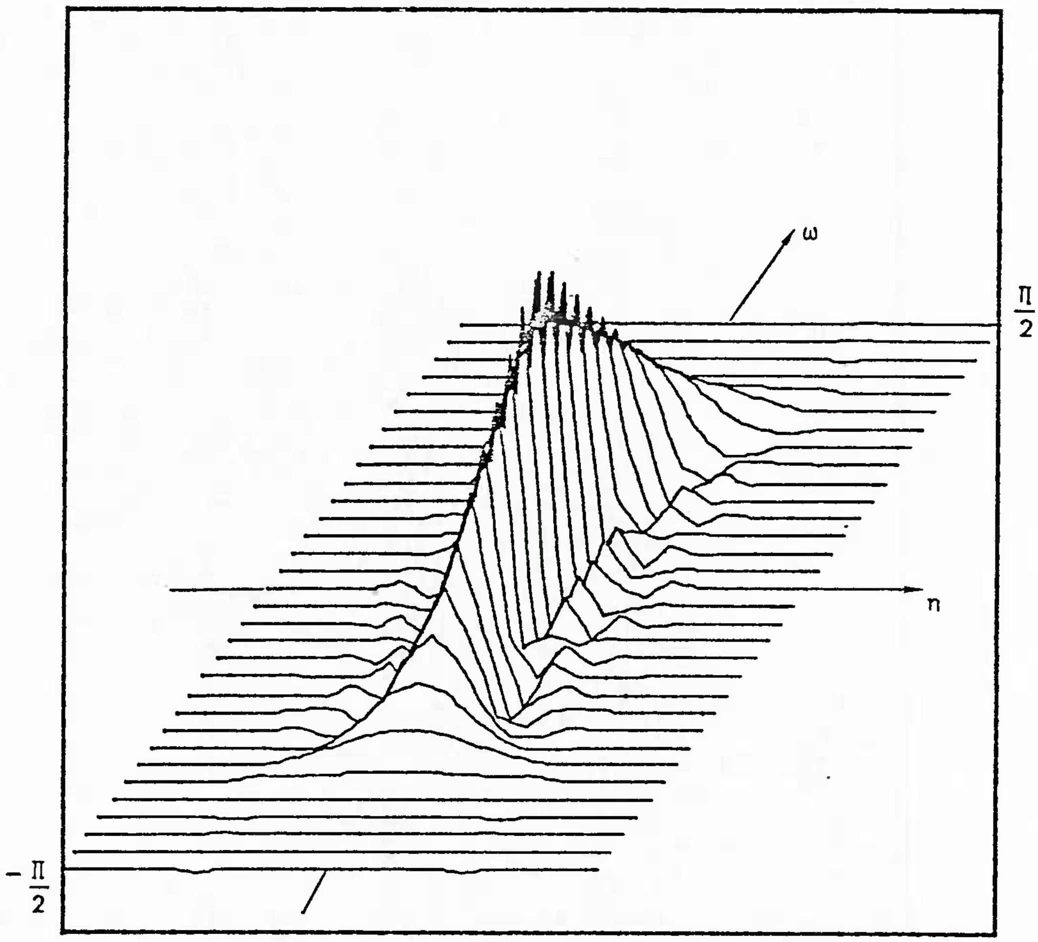

FIGURE 3

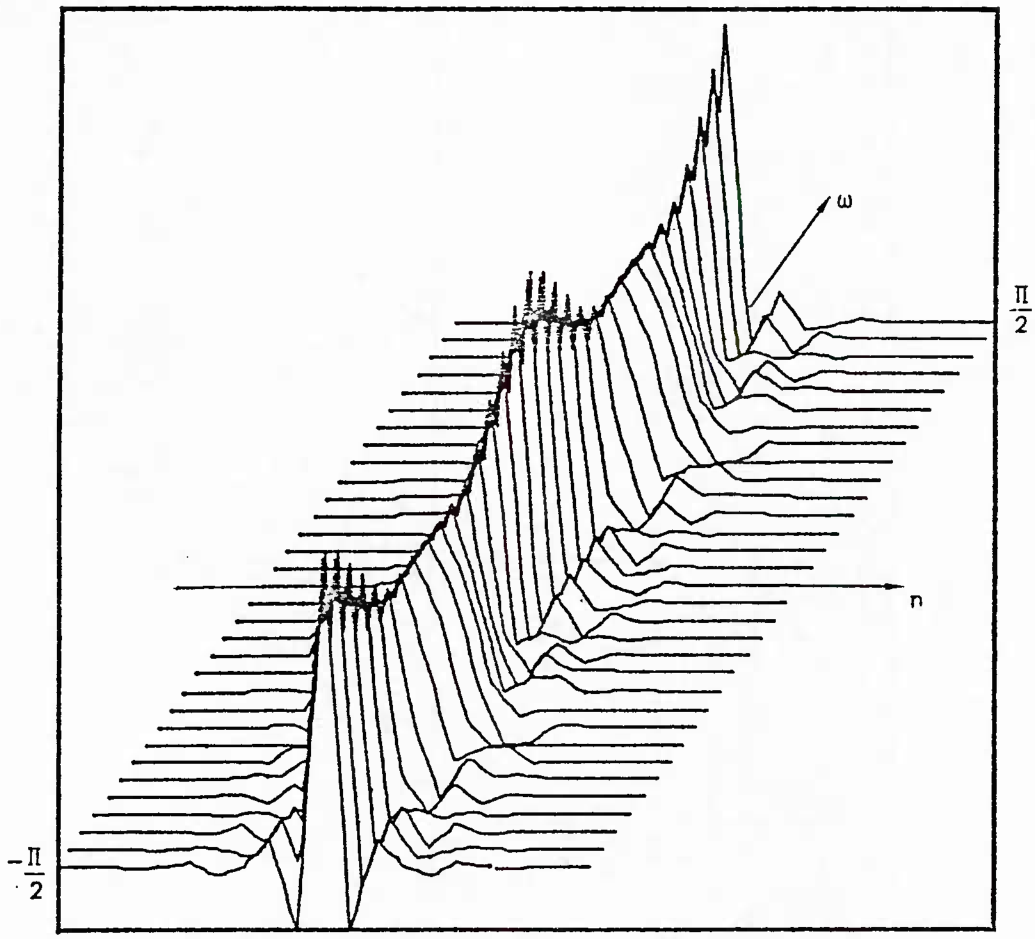

FIGURE 4

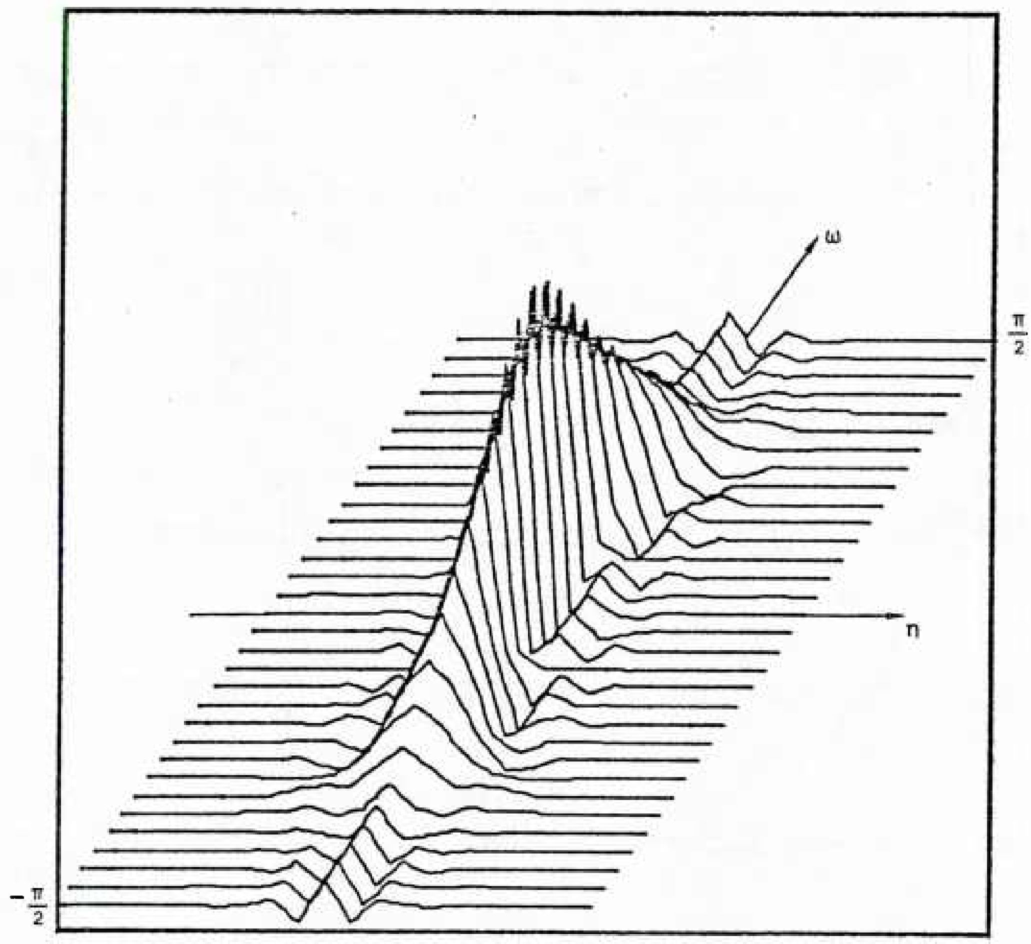

FIGURE 5